Tentamen Dynamische Systemen-2004

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For **each** question, please give **a precise and well constructed answer.**
You are kindly invited to pay attention on the style of the redaction. This will help the correctors.

Please: do not forget to indicate your, FAMILY NAME, first name, <student number> ... and your email adress.

Excercise 1 Let

$$P : \mathcal{K} \subset \mathbb{R} \to \mathbb{R}, x \mapsto x + \delta(x)$$

be a smooth real function where $\mathcal{K}$ is a compact interval and

$$\|\delta(x)\|_1 \leq 2$$

where $\|\cdot\|$ denote the the C^1 norm on the space of C^1 real valued functions, i.e., for each C^1 function f

$$\|f\|_1 = \sup_{x \in \mathcal{K}} \{\max\{|f(x)|, |f'(x)\ \}\}$$

[1.1] Show that x_0 is a non hyperbolic fixed point of P if and only if

$$\delta(x_0) = \delta'(x_0) = 0.$$

[1.2] Assume now that

$$\delta(x) = \gamma_0 + \gamma_1 x + x^3.$$

Draw the bifurcation set

$$\mathcal{B} = \{(\gamma_0, \gamma_1) \in \mathbb{R}^2 \mid \exists \tilde{x} \in \mathbb{R} \text{ such that, } \delta(\tilde{x}) = \delta'(\tilde{x}) = 0\}$$

[1.3] Show that this bifurcation set splits $\mathbb{R}^2$ into two connected componants, G_1 and G_2 such that

•$\forall(\gamma_0, \gamma_1) \in G_1,\ P$ possesses exaclty one hyperbolic fixed point,

•$\forall(\gamma_0, \gamma_1) \in G_2,\ P$ possesses exaclty three hyperbolic fixed points.

Problem: Consider the map

$$G : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto \frac{y^2}{2} + \frac{x^4}{4} - \frac{x^2}{2}$$

- Find the singular point of G, i.e., the set of points where dG is not submersive and the singular values.
- Show that or $E \notin \{0, -\frac{1}{4}\}$ $G^{-1}(E)$ is either
  - empty,
  - a closed curve,
  - two disjoint closed curve,
- Study the case $E = -\frac{1}{4}$ and $E = 0$. Give a draw of $G^{-1}(0)$.

Consider the following system

$$\mathcal{X} : \begin{cases} \dot{x} &= -y - G(x,y) \cdot (x^3 - x) \\ \dot{y} &= -x + x^3 - yG(x,y) \end{cases} \tag{1}$$

Denote by $\mathcal{X}_t$ the flow associated with $\mathcal{X}$.

[2.1] Find all the singularities of (1)

[2.2] Determine their type, i.e., saddle/sink/scource/center/..etc..

[2.3] Show that $\mathcal{X}_t$ leaves $G^{-1}(0)$ invariant. What is $G^{-1}(0)\setminus\{0\}$?

[2.4] Show that for all $-\frac{1}{4} < -b < 0 < a$, the flow $\mathcal{X}_t$ enters the set $G^{-1}\{-b < x < a\}$ and exits out of $G^{-1}\{-\frac{1}{4} < x < -b\}$.

[2.4] Show that $\mathcal{X}$ does not possess any periodic orbit.

We admit the following property to hold: $\forall \mathcal{U}$ neighborhood of $G^{-1}(0)$, $\exists \varepsilon > 0$ such that $G^{-1}[-\varepsilon, \varepsilon] \subset \mathcal{U}$.

[2.5] Take $p \notin G^{-1}\{0, -\frac{1}{4}\}$. Show that for all $\varepsilon > 0$ $\exists t \in \mathbb{R}$ such that $\mathcal{X}^t(p) \in G^{-1}[-\varepsilon, \varepsilon]$. Show that $\Omega(p)$ is either the union of a homoclinic orbit and a saddle point or the union of two homoclinic orbits and a saddle point.

Excercise 2 Consider the following 2 dimensional ordinary differential equations

$$\begin{cases} \dot{x} &= f(x, y; \gamma) = f_\gamma(x, y) \\ \dot{y} &= g(x, y; \gamma) = g_\gamma(x, y) \end{cases} \tag{2}$$

where γ is a parameter, $f, g : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ are smooth germs such that

- $f(0, 0; 0) = 0 = g(0, 0; 0)$

- f is a submersion and $\dfrac{\partial g}{\partial y}(0,0;0)\neq 0$.
- $\ker\, df_0(0,0) \equiv \ker\, dg_0(0,0)$.

[3.1] Show that for $\gamma = 0$, $(0,0)$ is a non hyperbolic singularity. We plan to study the bifurcation around this point.

[3.2] Find a germ of diffeomorphism of the form

$$\varphi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0),\ (x, y; \gamma) \mapsto (\varphi_{x,\gamma}(x, y; \gamma), \varphi_{y,\gamma}(x, y; \gamma); \gamma) = (X, Y; \gamma)$$

such that

$$\varphi\{(x, y; \gamma) \mid g_y(x, y; \gamma) = 0\} = \{Y = 0\}$$

Denote by

$$\hat{f}_\gamma(x, y) = \hat{f}(X, Y; \gamma) = f \circ \varphi^{-1}(X, Y; \gamma)$$

[3.3] Show that $\dfrac{\partial \hat{f}_0}{\partial X}(0,0) = 0$ and that the set of singularity is given by

$$\mathcal{S} = \{(X, Y, \gamma) \mid \hat{f}_\gamma(X, Y) = 0, Y = 0\}$$

[3.4] Assume $\dfrac{\partial^2 \hat{f}_0}{\partial X^2}(0,0)\neq 0$. Show that there exists $\tilde{X}(\gamma)$ depending smoothly in γ such that

$$\frac{\partial \hat{f}_\gamma}{\partial X}(\tilde{X}(\gamma), 0) \equiv 0,\ \tilde{X}(0) = 0.$$

Assume finally that $\dfrac{\partial \hat{f}}{\partial \gamma}(0,0;0)\neq 0$. Put $X = u + \tilde{X}(\gamma)$. Conclude that

$$\mathcal{S} = \{(\tilde{X}(\gamma) + u, Y; \gamma) \mid h(u; \gamma) = 0, Y = 0\}$$

where

$$h(u; \gamma) = h_\gamma(u) = \hat{f}(u + \tilde{X}(\gamma), 0; \gamma)$$

Write the asymtotic of h_γ. Deduce a description of the bifurcation diagram for singularities.